On some Möbius transformations generating free semigroup

Piotr Słanina

Abstract. A complex number λ is called a "semigroup free" if the semigroup generated by matrices $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\lambda = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$ is free. In the paper we give a short survey of the results about domains of semigroup free points on the complex plane. We make also a graphical visualisation of this domain.

Keywords: free semigroups, semigroups of matrices.

2010 Mathematics Subject Classification: 20M05, 20E05.

1. Introduction

Let λ be any complex number and let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B_\lambda = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}, C_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are the matrices from $GL(2, \mathbb{C})$. A number λ is called a "free point" or "free" if $gp(A, B_\lambda)$ – the subgroup generated by A and B_λ is free (otherwise it is called a "nonfree point"). If $sgp(A, B_\lambda)$ – the semigroup generated by A and B_λ is a free semigroup then λ is called a "semigroup free point" or "semigroup free" (otherwise it is called a "semigroup nonfree point").

The natural question is which complex numbers are free (or semigroup free). There are two main approaches to that question:

P. Słanina
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland,
e-mail: piotr@slanina.com.pl

R. Wituła, D. Słota, W. Hołubowski (eds.), *Monograph on the Occasion of 100th Birthday Anniversary of Zygmunt Zahorski*. Wydawnictwo Politechniki Śląskiej, Gliwice 2015, pp. 313–323.

(a) enlarging the set of known free points which form an open set (see Figure 1 – the set outside the black area) – the most important results are presented in [23, 3, 5, 20, 10, 16, 17, 26];
(b) determining new families of nonfree points (for example [25, 2, 8, 13, 21, 18, 14, 15, 22, 6, 12]).

We recall some facts proved in [27], Proposition 2.1.

Property 1.1.

(i) Let $A_1, A_2, \ldots$ be any square matrices of the same order over the same ring. If the group $gp(A_1, A_2, \ldots)$ is free, then the semigroup $sgp(A_1, A_2, \ldots)$ is free.
(ii) Let $2\lambda = \nu\mu$. Then the semigroup $sgp(A, B_\lambda)$ is free if and only if $sgp(B_\mu, C_\nu)$ is free. The same fact holds for the groups $gp(A, B_\lambda)$ and $gp(B_\mu, C_\nu)$
(iii) The semigroup $sgp(B_\mu, C_\mu)$ is free if and only if $sgp(B_\mu, J)$ is a free product of cyclic semigroups generated by B_μ and J. The same property holds for groups $gp(B_\mu, C_\mu)$ and $gp(B_\mu, J)$.
(iv) Every transcendental λ is free (and hence semigroup free) [7].

Note that there is an isomorphism between $PSL(2, \mathbb{C})$ and the group of homographic functions namely

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(z \to \frac{az+b}{cz+d}\right).$$

Let

$$\alpha = (x \mapsto x+2)\,,\ \beta_\lambda = \left(x \mapsto \frac{x}{\lambda x+1}\right),\ \gamma_\lambda = (x \mapsto x+\lambda)\,,\ \iota = \left(x \mapsto \frac{1}{x}\right).$$

Then

$$\varphi(A) = \alpha, \varphi(B_\lambda) = \beta_\lambda,\ \varphi(C_\lambda) = \gamma_\lambda,\ \varphi(J) = \iota,\ \beta_\lambda = \iota\gamma_\lambda\iota.$$

Then $sgp(A, B_\lambda)$ is free if and only if $sgp(\alpha, \beta_\lambda)$ is free.

Further we will consider homographic functions instead of matrices.

Let G be any subgroup of $PSL(2, \mathbb{C})$. By [24], Definition 2.1.1, an elements of a group G form a normal family in a domain $\Omega(G) \subset \mathbb{C}$ if every sequence of elements $\{g_n\} \subset G$ contains either a subsequence which converges to a limit element $g \not\equiv \infty$ uniformly on each compact subset of $\Omega(G)$, or a subsequence which converges uniformly to ∞ on each compact subset.

If G is discrete then the set $\Omega(G)$ on which the elements form a normal family is called the regular set of G. For these groups, the set of λ's, for which $\Omega(gp(\alpha, \beta_\lambda))/gp(\alpha, \beta_\lambda)$ is a four times punctured sphere is called *The Riley slice of Schottky space* and consists of free points [19].

In the Figure 1 (due to David Wright, see for example [19]), the Riley slice of Schottky space is the set of points lying outside the black area. Thus the complement of the Riley slice is the black quasi-rhombus which has vertices at the points $2, -2, i, -i$. This Figure includes all information about free points we can get from publications mentioned in (a). However, there are no papers describing the algorithm of this fractal creation.

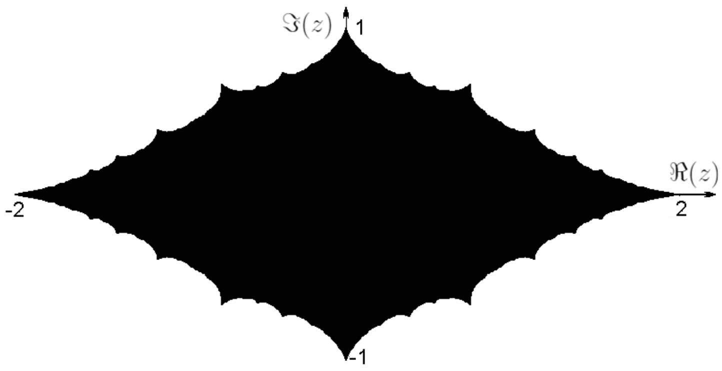

Fig. 1

Our aim is to collect all known facts about semigroup free points. In contrast to free points, semigroup free points are not so recognized and to prove that some complex numbers are semigroup free or not, we need often different and more complicated methods. For example, for $gp(A, B_\lambda)$ to be nonfree we only need to find any nontrivial word $W(A, B_\lambda)$ which is equal to identity while $sgp(A, B_\lambda)$ is nonfree if and only if either there exists any nontrivial word $W(A, B_\lambda)$ with positive exponents which is equal to identity matrix or if two different words with positive exponents are equal.

2. Semigroup free points

The first result was Theorem 2.6 in [4] and it can be described by Figure 2, where points outside the grey area are semigroup free.

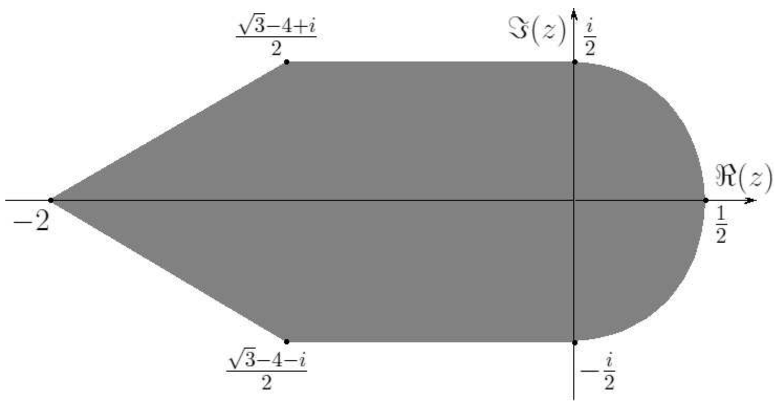

Fig. 2

Bamberg used a computer program to find some free points and showed his result in the Figure 4 in [1]. In this figure, white points outside the black area represent free points, white points inside the black area represent nonfree points. The border of the black area consists of some arcs and lines based on [5, 20, 9, 11]. Thanks to the picture with the Riley slice of Schottky space (which consists of free points) from [19] and Proposition 1.1 (i), the border in the Figure 4 in [1] can be improved. In [27], Corollary 3.6, the author presents another picture (see Figure 3) based on Figure 2, Figure 4 in [1] and Corollary 3.2 in [27]:

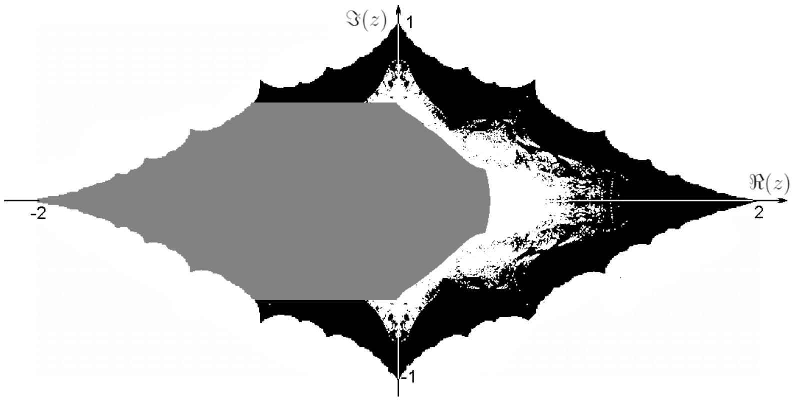

Fig. 3

Property 2.1. *In the Figure 3:*

(i) White points outside the grey and black areas are free.
(ii) White points inside the black area mark area where the set of nonfree points is "almost dense" [1] *and these points are semigroup free at the same time.*
(iii) All points outside the grey area are semigroup free.

We will show that the set of known semigroup free points can be enlarged. First we recall some known results.

Property 2.2. *([27], Proposition 2.2) Let* $a_1, a_2, b_1, b_2 \in \mathbb{R}$*,* $b_1 \neq 0$ *and* $b_2 \neq 0$*. If*

$$z = a_1 + a_2 i + (b_1 + b_2 i)t, \ t \in \mathbb{R}$$

is the parametric equation of a line in the complex plane $\mathbb{C}$*, not including the origin, then the transformation* $\iota : z \longrightarrow \frac{1}{z}$ *maps this line into a circumference (which includes origin) defined as*

$$\left| z + \frac{b_2 + b_1 i}{2(a_2 b_1 - a_1 b_2)} \right| = \frac{|b_1 + b_2 i|}{2|a_2 b_1 - a_1 b_2|}.$$

[1] The area of nonfree points is called "almost dense" if every pixel in this area contains a nonfree point [1].

By Lemma 2.5 and Corollary 2.6 in [27], we have:

Lemma 2.3. *Let G be a group which acts on the set X and let H_1, H_2 be infinite cyclic semigroups of the group G. Let X_1, X_2 be two nonempty disjoint subsets of the set X such that*

(i) for every $h_1 \in H_1$, $h_1(X_1 \cup X_2) \subset X_2$,
(ii) for every $h_2 \in H_2$, $h_2(X_2) \subset X_1$.

Then the semigroup generated by H_1 and H_2 is free.

Let $\lambda = \lambda_1 + \lambda_2 i$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ and let

$$X_1 = \{z \in \mathbb{C} : \lambda_2 Re(z) < -\lambda_1 Im(z) < \lambda_2 Re(z-2) \quad \wedge \quad Im(z) > 0\},$$

$$X_2 = \{z \in \mathbb{C} : \lambda_2 Re(z-2) < -\lambda_1 Im(z) \quad \wedge \quad Im(z) > 0\},$$

$$H_1 = gp\left\langle \alpha_\lambda \right\rangle, \quad H_2 = gp\left\langle \beta_\lambda \right\rangle.$$

Then for every natural m, n:

$$\alpha^m(X_1) =$$
$$= \{z : \lambda_2 Re(z-2m) < -\lambda_1 Im(z-2m) < \lambda_2 Re(z-2m-2) \quad \wedge \quad Im(z) > 0\} \subset X_2$$

and $\alpha^m(X_2) \subset X_2$. Without loss of generality, we assume that $\lambda_2 < 0$. The equality $\lambda_2 Re(z-2) = -\lambda_1 Im(z)$ defines a line

$$z = 1 + \frac{\lambda_2}{\lambda_1} i + (\lambda_2 - \lambda_1 i)t, \ t \in \mathbb{R}.$$

Hence by Property 2.2, $\iota(X_2)$ is an intersection of the circle

$$\left| z - \frac{1}{4} + \frac{\lambda_1}{4\lambda_2} i \right| < \frac{\sqrt{\lambda_1^2 + \lambda_2^2}}{-4\lambda_2}$$

and the half-plane $\{z \in \mathbb{C} : Im(z) < 0\}$. To simplify the notation, let

$$r(\iota(X_2)) = \frac{\sqrt{\lambda_1^2 + \lambda_2^2}}{-4\lambda_2}.$$

The sets $\iota\gamma\lambda^n(X_2)$ are tangent to the line $\lambda_2 Re(z) = -\lambda_1 Im(z)$. Because the vector $[\lambda_1, \lambda_2]$ is parallel to the line $\lambda_2 Re(z) = -\lambda_1 Im(z)$, the sets $\iota\gamma\lambda^n(X_2)$ are tangent to this line for any natural n and $z \in \iota\gamma\lambda^n(X_2)$ implies $Im(z) < 0$.

We will find the conditions implying

$$\iota\gamma\lambda(X_2) \cap \iota(X_2) = \varnothing \tag{1}$$

(i) Let $\lambda_2 \geqslant \lambda_1$ (see Figure 4). Then (1) holds if $2r(\iota(X_2)) \leqslant |\lambda_1 + \lambda_2 i|$ and hence $-\frac{1}{2} \geqslant \lambda_2$ (observe that $-\frac{1}{2} \geqslant \lambda_2 \geqslant \lambda_1$ imply $8\lambda_1 \geqslant 1 - 16\lambda_2^2$).
(ii) Let $\lambda_1 \geqslant 0$ (see Figure 5). Then (1) holds if

$$|\lambda_2| \geqslant r(\iota(X_2)) - \sqrt{r^2(\iota(X_2)) - \frac{1}{16}}.$$

(iii) Let $\lambda_2 \leqslant \lambda_1 \leqslant 0$ (see Figure 5). Then (1) holds if

$$|\lambda_2| \geqslant r(\iota(X_2)) + \sqrt{r^2(\iota(X_2)) - \frac{1}{16}}.$$

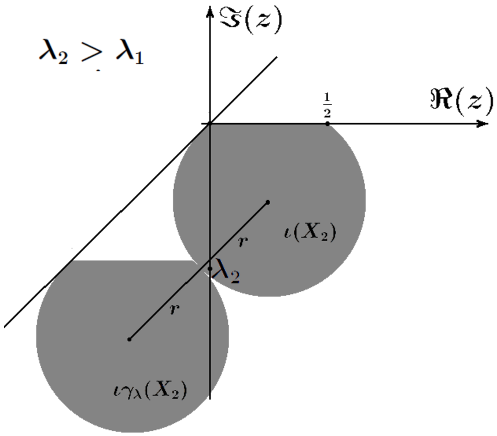

Fig. 4

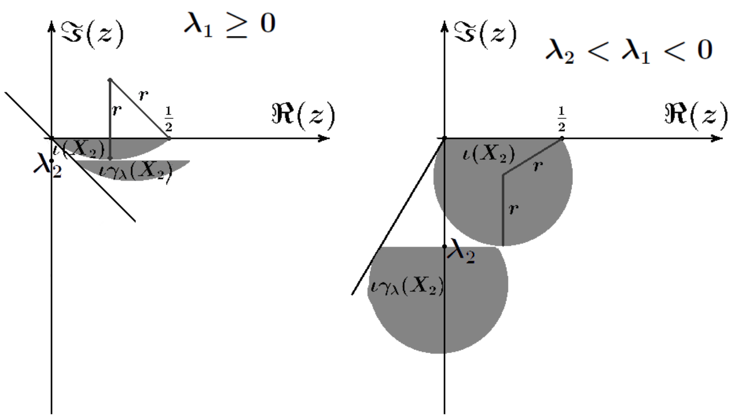

Fig. 5

Each of the two last inequalities is equivalent to

$$4\lambda_2^2 + \lambda_1 \geqslant \sqrt{\lambda_1^2 + \lambda_2^2}. \tag{2}$$

If $4\lambda_2^2 + \lambda_1 < 0$ then inequality (2) is false so let $4\lambda_2^2 + \lambda_1 \geqslant 0$. Then the inequality (2) can be transformed to

$$8\lambda_1 \geqslant 1 - 16\lambda_2^2 \tag{3}$$

(observe that $8\lambda_1 \geqslant 1-16\lambda_2^2$ implies $4\lambda_2^2+\lambda_1 \geqslant 0$). Hence (3) implies $\iota\gamma_\lambda^n(X_2)\cap\iota(X_2) = \varnothing$ and $\beta_\lambda^n(X_2) = \iota\gamma_\lambda^n\iota^n(X_2) \subset X_1$ for any natural n and hence by Lemma 2.3, λ is semigroup free.

If we assume that $\lambda_2 = Im(\lambda) > 0$ then we also get (3).

All the above considerations give us the following

Theorem 2.4. *If* $8\lambda_1 \geqslant 1 - 16\lambda_2^2$ *and* $\lambda_2 \neq 0$ *then* $\lambda = \lambda_1 + \lambda_2 i$ *is a semigroup free point.*

We present results from Theorem 2.4, Figure 1 and [4] (Theorems 2.5 and 2.6) at the Figure 6.

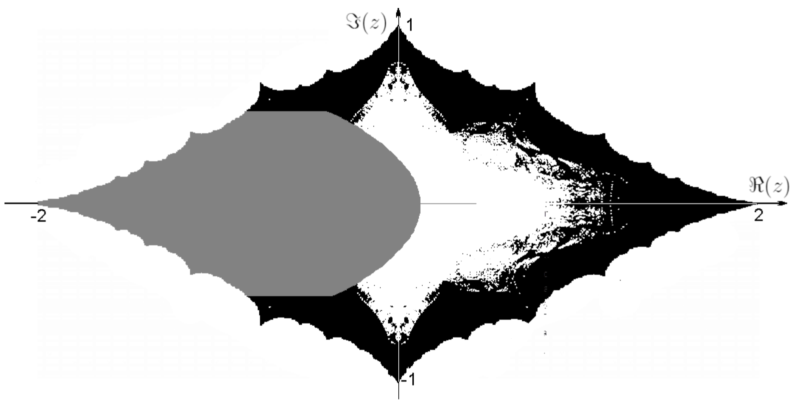

Fig. 6

Theorem 2.5. *Points outside the grey area at the Figure 6 are semigroup free points.*

3. Semigroup nonfree points

To show that $sgp(A, B_\lambda)$ is nonfree it suffices to find two different words $W_1(A, B_\lambda)$ and $W_2(A, B_\lambda)$ (one of them can be empty) such that $W_1(A, B_\lambda) = W_2(A, B_\lambda)$. Note that the shortest nontrivial relations are of the form

$$A^{k_1}B_\lambda^{k_2}A^{k_3} = B_\lambda^{k_4}A^{k_5}B_\lambda^{k_6}$$

and if

$$\lambda = \frac{1}{2}\left(\frac{1}{k_3k_6} - \frac{1}{k_4k_5} - \frac{1}{k_5k_6}\right), \quad k_1 = \frac{k_5k_6}{k_2}, \quad k_2 = \frac{k_4k_5}{k_3} \tag{4}$$

for any natural $k_1, \cdots, k_6$ then λ is semigroup nonfree point (compare [4] Theorem 3.04).

Property 3.1. *The set of λ satisfying (4) has limit points -1, $\frac{1}{2}$, $\frac{1}{2a}$, $-\frac{1}{2a} - \frac{1}{2b}$ and $\frac{1}{2a} - \frac{1}{2b}$ for any positive integer a and b.*

Proof. The fact that the numbers -1, $\frac{1}{2}$ are limits of nonfree points follows from [4], Theorem 3.04.

If $k_6 = k_4 = 1$ and $k_3 = a$ then

$$\lim_{k_5\to\infty} \frac{1}{2}\left(a^{-1} - k_5^{-1} - k_5^{-1}\right) = \frac{1}{2a}.$$

If $k_5 = 1$, $k_4 = a$ and $k_6 = b$ then

$$\lim_{k_3\to\infty} \frac{1}{2}\left((k_3b)^{-1} - a^{-1} - b^{-1}\right) = -\frac{1}{2a} - \frac{1}{2b}.$$

If $k_6 = 1$, $k_3 = a$ and $k_5 = b$ then

$$\lim_{k_4\to\infty} \frac{1}{2}\left(a^{-1} - (k_4b)^{-1} - b^{-1}\right) = \frac{1}{2a} - \frac{1}{2b}.$$

□

In [4], Brenner and Charnov investigated mainly in the semigroup $sgp(C_\mu, B_\mu)$; we reformulate Theorems 4.1–4.6, 5.1, 5.2 and 6.1 from [4] to the semigroup $sgp(A, B_\lambda)$; the reason is to collect finally all results concerning semigroup free and nonfree points in the common picture.

Theorem 3.2.

(1) The nonfree points λ are dense on $[-2, 0]$ and are arbitrarily close to $\frac{1}{2}$.
(2) If the word $W(A, B_\lambda)$ has finite order and length not greater than 4 then λ is real and negative.
(3) Let n be a nonzero integer. Then $sgp(A, B_\lambda)$ has torsion element for $\lambda \in \{-\frac{1}{2n^2}, -\frac{1}{n^2}, -\frac{3}{2n^2}\}$.
(4) If λ is real and positive then $sgp(A, B_\lambda)$ is torsion free.
(5) If $\lambda = -\frac{p^2}{2q^2}$, p and r are nonzero integers and $p \neq \pm 1$, $gcd(p, q) = 1$ then $gp(A, B_\lambda)$ (and hence $sgp(A, B_\lambda)$) is torsion free.
(6) Let n be an integer. Then $sgp(A, B_\lambda)$ is not free if $\lambda = \pm\frac{8}{n^2}$ for $n > 4$.
(7) Let $\sqrt{2\lambda}$ be a primitive r-th root of 1. Then $sgp(A, B_\lambda)$ is free if and only if $r \neq 4$.

Fact given in Theorem 3.2 (7) was proved in [4], Theorem 6.1, for all the cases except $q \in \{3, 6\}$. Observe that $(\cos(2\pi/3) + i\sin(2\pi/3))/2$ and $(\cos(4\pi/3) + i\sin(4\pi/3))/2$ are free points by Theorem 2.4, because the parabola $8\lambda_1 = 1 - 16\lambda_2^2$ includes such λ's.

The following equality

$$A^{k_1} B_\lambda^{k_2} A^{k_3} B_\lambda^{k_4} A^{k_5} B_\lambda^{k_6} = B_\lambda^{k_6} A^{k_5} B_\lambda^{k_4} A^{k_3} B_\lambda^{k_2} A^{k_1}$$

holds if and only if

$$\begin{aligned}
&4k_1k_2k_3k_4k_5k_6\,\lambda^2 + \\
&\quad + 2\big(k_1k_2k_3k_4 + k_3k_4k_5k_6 + k_1k_2k_5k_6 + k_1k_4k_5k_6 + k_1k_2k_3k_6 - k_2k_3k_4k_5\big)\,\lambda + \\
&\qquad + k_3k_4 + k_1k_2 + k_1k_4 + k_5k_6 + k_1k_6 + k_3k_6 - k_2k_5 - k_4k_5 - k_2k_3 = 0
\end{aligned} \tag{5}$$

and the equality

$$A^{k_1} B_\lambda^{k_2} A^{k_3} B_\lambda^{k_4} A^{k_5} B_\lambda^{k_6} A^{k_7} B_\lambda^{k_8} = B_\lambda^{k_8} A^{k_7} B_\lambda^{k_6} A^{k_5} B_\lambda^{k_4} A^{k_3} B_\lambda^{k_2} A^{k_1}$$

holds if and only if

$$\begin{aligned}
&8k_1k_2k_3k_4k_5k_6k_7k_8\lambda^3 + \\
&\quad + 4\big(k_5k_6k_7k_8k_1k_4 + k_5k_6k_7k_8k_3k_4 + k_5k_6k_7k_8k_1k_2 + k_5k_6k_1k_2k_3k_4 + \\
&\quad + k_7k_8k_1k_2k_3k_4 + k_5k_8k_1k_2k_3k_4 + k_6k_7k_8k_1k_2k_3 - k_5k_6k_7k_2k_3k_4\big)\,\lambda^2 + \\
&\quad + 2\big(k_5k_6k_7k_8 + k_5k_6k_1k_2 + k_5k_6k_3k_4 + k_5k_6k_1k_4 + k_7k_8k_1k_2 + k_7k_8k_3k_4 + \\
&\quad + k_7k_8k_1k_4 + k_1k_2k_3k_4 + k_5k_8k_1k_2 + k_5k_8k_3k_4 + k_5k_8k_1k_4 + k_8k_1k_2k_3 + \\
&\quad + k_6k_1k_2k_3 + k_6k_7k_8k_1 + k_6k_7k_8k_3 - k_7k_3k_4k_2 - k_5k_2k_3k_4 - k_5k_6k_7k_2 - \\
&\quad - k_5k_6k_7k_4 - k_6k_7k_3k_2\big)\,\lambda + k_8k_1 + k_8k_3 + k_6k_1 + k_6k_3 + k_1k_2 + k_3k_4 + k_1k_4 + \\
&\qquad + k_5k_8 + k_7k_8 + k_5k_6 - k_7k_2 - k_7k_4 - k_5k_2 - k_5k_4 - k_6k_7 - k_2k_3 = 0.
\end{aligned} \tag{6}$$

We have used a computer program to make a graphical visualisation of nonfree points based on Figure 6 and then we get Figure 7. Light grey points are roots of the polynomial (5), evaluated for $k_1, k_4, k_5, k_6 \in \{1, \ldots, 20\}$ and $k_2, k_3 \in \{1, \ldots, 200\}$ and roots of the polynomial (6) for $k_1, \ldots, k_8 \in \{1, \ldots, 13\}$ and for $k_1, k_4, k_5, k_8 \in \{1, \ldots, 4\}$ and $k_2, k_3, k_6, k_7 \in \{1, \ldots, 50\}$.

Property 3.3. *In the Figure 7:*

(1) Light grey points mark area where the set of semigroup nonfree points is "almost dense".
(2) White points inside the quasi-rhombus mark area where the set of nonfree points is "almost dense". All this points are semigroup free.
(3) Points outside the grey area are semigroup free.
(4) White points outside the quasi-rhombus are free.

There are still much of complex numbers we don't know if they are semigroup free or not – for example, some algebraic numbers from the quasi-rhombus in the Figure 7.

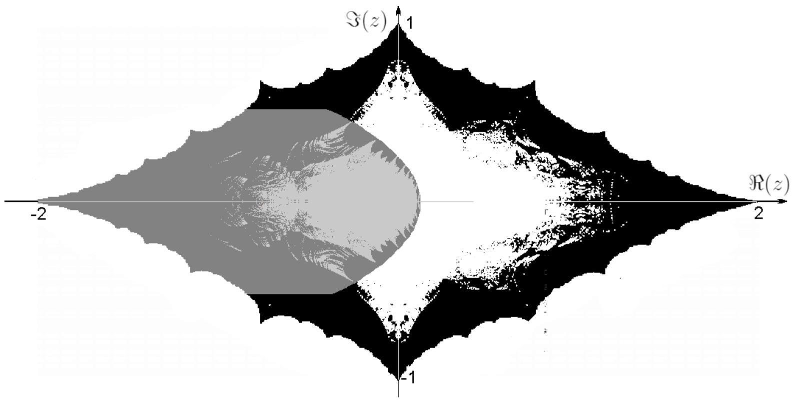

Fig. 7

Bibliography

1. Bamberg J.: *Non-free points for groups generated by a pair of* 2×2 *matrices.* Proc. London Math. Soc. II **62** (2000), 795–801.
2. Beardon A.-F.: *Pell's equation and two generator free Moebius groups.* London Math. Soc. **25** (1993), 527–532.
3. Brenner J.-L.: *Quelques groupes libres de matrices.* C. R. Acad. Sci. Paris **241** (1955), 1689–1691.
4. Brenner J.-L., Charnow A.: *Free semigroups of 2 × 2 matrices.* Pacific J. Math. **77** (1978), 57–69.
5. Chang B., Jennings S.-A., Ree R.: *On certain matrices which generate free groups.* Canadian J. Math. **10** (1958), 279–284.
6. Evans R.-J.: *Non-free groups generated by two parabolic matrices.* J. Res. Nat. Bur. Stand. **84** (1979), 179–180.
7. Fouxe-Rabinowitch D.-I.: *On a certain representation of a free group.* Leningrad State Univ. Annals (Uchennye Zapiski). Math. Ser. **10** (1940), 154–157.
8. Grytczuk A., Wojtowicz M.: *The problem of freeness for Euler monoids and Moebius groups.* Semigroup Forum **61** (2000), 277–282.
9. Ignatov Yu.-A.: *Free groups generated by two parabolic-fractional linear transformations.* Modern Algebra **4** (1976), 87–90.
10. Ignatov Yu.-A.: *Free and nonfree subgroups of* $\mathrm{PSL}_2(\mathbb{C})$ *that are generated by two parabolic elements.* Mat. Sb. (N.S.) **106 (148)** (1978), 372–379 (in Russian).
11. Ignatov Yu.-A.: *Free and nonfree subgroups of* $\mathrm{PSL}_2(\mathbb{C})$ *that are generated by two parabolic elements.* Mat. Sb. **35** (1979), 49–55 (in Russian).
12. Ignatov Yu.-A.: *Roots of unity as nonfree points of the complex plane.* Mat. Zametki **27** (1980), 825–827.
13. Ignatov Yu.-A.: *Rational nonfree points of the complex plane.* Tulsk. Gos. Ped. Inst., Tula 1986, 72–80.
14. Ignatov Yu.-A.: *Rational nonfree points of the complex plane.* Tulsk. Gos. Ped. Inst., Tula 1990, 53–59.
15. Ignatov Yu.-A.: *Rational nonfree points of the complex plane, part III.* Tulsk. Gos. Ped. Inst., Tula 1995, 78–84.
16. Ignatov Yu.-A., Gruzdeva T.-N., Sviridova I.-A.: *Free groups of linear-fractional transformations.* Tulsk. Gos. Univ. Ser. Mat. Mekh. Inform. **5** (1999), 116–120 (in Russian).
17. Ignatov Yu.-A., Evtikhova A.V.: *Free groups of linear-fractional transformations.* Chebyshevskii Sb. **3** (2002), 78–81 (in Russian).
18. Kabieniuk M.-I.: *Groups Generated by Two Transitive Matrices.* Kemerowskij Gos. Univ., Kemerovo 1988.

19. Keen L., Series C.: *The Riley slice of Schottky space*. Proc. London Math. Soc. II **69** (1994), 72–90.
20. Lyndon R.-C., Ullman J.-L.: *Groups generated by two parabolic linear fractional transformations*. Canadian J. Math. **21** (1969), 1388–1403.
21. Lysenkov S.-W.: *Groups generated by two* 2×2 *matrices*. Kemerowskij Gos. Univ. (1983), 27–31.
22. Newman M.: *A conjecture on a matrix group with two genetators*. J. Res. Nat. Bur. Standards. **78B** (1974), 69–70.
23. Sanov I.-N.: *A property of a representation of a free group*. DAN CCCP **57** (1947), 657–659 (in Russian).
24. Schiff J.-L.: *Normal Families*. Springer-Verlag, New York 1993.
25. Skuratskii A.-I.: *The problem of the generation of free groups by two unitriangular matrices*. Mat. Zametki **24** (1978), 411–414.
26. Słanina P.: *On some free groups, generated by matrices*. Demonstratio Math. **37** (2004), 55–61.
27. Słanina P.: *On some free semigroups, generated by matrices*. Czech. Math. J. (in press).